

ON THE QUANTITATIVE DYNAMICAL MORDELL-LANG CONJECTURE

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ABSTRACT. The dynamical Mordell-Lang conjecture concerns the structure of the intersection of an orbit in an algebraic dynamical system and an algebraic variety. In this paper, we bound the size of this intersection for various cases when it is finite.

1. INTRODUCTION

1.1. Motivation. Let \mathcal{X} be an algebraic variety defined over the complex numbers \mathbb{C} , and let $\Phi : \mathcal{X} \rightarrow \mathcal{X}$ be a morphism. For any integer $n \geq 0$, we denote by $\Phi^{(n)}$ the n -th iteration of Φ with $\Phi^{(0)}$ denoting the identity map.

Throughout the paper, a single integer is viewed as an arithmetic progression with common difference 0.

The following is the well-known *dynamical Mordell-Lang conjecture* for self-morphisms of algebraic varieties in the dynamical setting; see [11, 16, 17].

Conjecture 1.1 (Dynamical Mordell-Lang Conjecture). *Let \mathcal{X} and Φ be given as the above, let $V \subseteq \mathcal{X}$ be a closed subvariety, and let $P \in \mathcal{X}(\mathbb{C})$. Then, the following subset of integers*

$$\{n \geq 0 : \Phi^{(n)}(P) \in V(\mathbb{C})\}$$

is a finite union of arithmetic progressions.

Conjecture 1.1 has been studied extensively in recent years. However, so far there are only a few related results. These include results on maps of various special types [4, 5, 7, 14, 16, 17, 23, 24] (especially diagonal maps), and analogues for Noetherian spaces [6] and Drinfeld modules [15].

Recently, Silverman and Viray [23, Corollary 1.4] have given results regarding the uniform boundedness (only in terms of m) of intersections

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of orbits of the power map (with the same exponent) at a point of the projective m -space $\mathbb{P}^m(\mathbb{C})$ with non-zero multiplicatively independent coordinates, with any linear subspace of $\mathbb{P}^m(\mathbb{C})$. However, they have not provided quantitative results. In fact, such a result follows, even in a more general case, directly from the uniform bound on the number of zeros of simple and non-degenerate linear recurrence sequences.

We also note that the uniform boundedness condition has recently been considered in [10], where several results are given for the frequency of the points in an orbit of an algebraic dynamical system that belong to a given algebraic variety under the reduction modulo a prime p .

1.2. Our Results. In this paper, we study the quantitative version of Conjecture 1.1 for polynomial morphisms of several special types when \mathcal{X} is the affine m -space $\mathbb{A}^m(\mathbb{C})$ and V is a hypersurface; see Section 3. Our main objective is to find as many classes of polynomial morphisms as possible having uniform bounds (or as close as possible to uniformity), and not to investigate detailedly the quality of the bounds. To the best of our knowledge, this is the first work on the quantitative dynamical Mordell-Lang conjecture.

Here, we extend the results of Silverman and Viray [23] in two directions. First, we consider monomial systems with different exponents. Second, we estimate the size of the intersection of an orbit with a hypersurface rather than with a hyperplane.

For example, we illustrate a typical case of our results; see Theorem 3.1 for more details. Let $\Phi = (X_1^d, \dots, X_m^d)$ with integer $d \geq 2$ be a diagonal endomorphism of $\mathbb{A}^m(\mathbb{C})$. Fix a hypersurface V , defined by a non-zero polynomial

$$G = \sum_{i_1, \dots, i_m} a_{i_1, \dots, i_m} X_1^{i_1} \cdots X_m^{i_m} \in \mathbb{C}[X_1, \dots, X_m].$$

Then, for any $\mathbf{w} \in (\mathbb{C}^*)^m$ with multiplicatively independent coordinates, the size of the intersection of V and the orbit of Φ at the point \mathbf{w} is at most

$$(8\mathbf{n}(G))^{4\mathbf{n}(G)^5},$$

where $\mathbf{n}(G)$ is the number of monomials of G .

Our methods rely on estimates (when finite) for integer solutions of certain polynomial-exponential equations. For the case of the power map studied by Silverman and Viray [23] we employ results on the number of zeros in linear recurrence sequences due to [1, 2] and [19]. For more general monomial systems we use results on the number of solutions in a finitely generated subgroup of $(\mathbb{C}^*)^k$ of linear equations of the form $a_1x_1 + \dots + a_kx_k = 0$, $a_1, \dots, a_k \in \mathbb{C}^*$, as well as solutions

to more general polynomial-exponential equations due to Schlickewei and Schmidt [20].

In fact, by [16, Theorem 1.8] the Dynamical Mordell-Lang Conjecture is known to hold in the cases we consider, because the morphisms can essentially restrict to endomorphisms of $(\mathbb{C}^*)^m$. Besides, the methods we use might be not applicable on other kinds of morphisms, see Section 4 for more details.

1.3. Convention and notation. For integer $m \geq 2$, let

$$\Phi = (F_1, \dots, F_m) : \mathbb{A}^m(\mathbb{C}) \rightarrow \mathbb{A}^m(\mathbb{C}), \quad F_1, \dots, F_m \in \mathbb{C}[X_1, \dots, X_m],$$

be a morphism defined by a system of m polynomials in m variables over \mathbb{C} . For each $i = 1, \dots, m$, we define the n -th iteration of the polynomials F_i by the recurrence relation

$$F_i^{(0)} = X_i, \quad F_i^{(n)} = F_i \left(F_1^{(n-1)}, \dots, F_m^{(n-1)} \right), \quad n = 1, 2, \dots,$$

so that

$$\Phi^{(n)} = \left(F_1^{(n)}, \dots, F_m^{(n)} \right).$$

See [3, 21, 22] for a background on dynamical systems associated with such iterations.

For a vector $\mathbf{w} = (w_1, \dots, w_m) \in \mathbb{C}^m$, we denote by

$$\text{Orb}_{\mathbf{w}}(\Phi) = \{ \Phi^{(n)}(\mathbf{w}) : n = 0, 1, 2, \dots \}$$

the orbit of Φ at \mathbf{w} . For an algebraic variety $V = Z(G_1, \dots, G_s)$ defined by the equations $G_1 = \dots = G_s = 0$, $G_i \in \mathbb{C}[X_1, \dots, X_m]$, $i = 1, \dots, s$, we consider the elements of the orbit $\text{Orb}_{\mathbf{w}}(\Phi)$ which fall into V and denote

$$(1.1) \quad \mathcal{S}_{\mathbf{w}}(\Phi, V) = \{ n \geq 0 : \Phi^{(n)}(\mathbf{w}) \in V \}.$$

We say that the complex numbers $\alpha_1, \dots, \alpha_n$ are *multiplicatively independent* if all of them are non-zero and there is no non-zero integer vector (i_1, \dots, i_n) such that $\alpha_1^{i_1} \dots \alpha_n^{i_n} = 1$.

In the sequel, we denote by $|S|$ the cardinality of a finite set S . Our objective in this paper is to bound the size of $|\mathcal{S}_{\mathbf{w}}(\Phi, V)|$ for various cases when it is finite.

Throughout the paper, let $\overline{\mathbb{Q}}$ be the algebraic closure of the rational numbers \mathbb{Q} . For any field K , we write K^* for the multiplicative group of all the non-zero elements of K . For any multiplicative group Λ and any integer $k \geq 1$, let Λ^k be the direct product consisting of k -tuples $\mathbf{x} = (x_1, \dots, x_k)$ with $x_i \in \Lambda$, $1 \leq i \leq k$. As usual, the multiplication of the group Λ^k is defined by $\mathbf{xy} = (x_1 y_1, \dots, x_k y_k)$ for any $\mathbf{x}, \mathbf{y} \in \Lambda^k$.

2. PRELIMINARIES

In this section, we gather some results which are used afterwards.

Recall that a linear recurrence sequence (LRS) of order $m \geq 1$ is a sequence $\{u_0, u_1, u_2, \dots\}$ with elements in \mathbb{C} satisfying a linear relation

$$(2.1) \quad u_{n+m} = a_1 u_{n+m-1} + \dots + a_m u_n \quad (n = 0, 1, 2, \dots),$$

where $a_1, \dots, a_m \in \mathbb{C}$, $a_m \neq 0$ and $u_j \neq 0$ for at least one j in the range $0 \leq j \leq m-1$. We assume that relation (2.1) is minimal, that is the sequence $\{u_n\}$ does not satisfy a relation of type (2.1) of smaller length.

The characteristic polynomial of this LRS $\{u_n\}$ is

$$f(X) = X^m - a_1 X^{m-1} - \dots - a_m = \prod_{i=1}^k (X - \alpha_i)^{e_i} \in \mathbb{C}[X]$$

with distinct $\alpha_1, \alpha_2, \dots, \alpha_k$ and $e_i > 0$ for $1 \leq i \leq k$. Then, u_n can be expressed as

$$u_n = \sum_{i=1}^k f_i(n) \alpha_i^n,$$

where f_i is some polynomial of degree $e_i - 1$ ($i = 1, 2, \dots, k$). We call the sequence $\{u_n\}$ *simple* if $k = m$ (that is $e_1 = \dots = e_m = 1$) and *non-degenerate* if α_i/α_j is not a root of unity for any $i \neq j$ with $1 \leq i, j \leq k$.

One fundamental problem of the LRS (2.1) is to describe the structure or bound the size of the following set

$$\{n \geq 0 : u_n = 0\},$$

which is called the *zero set* of the sequence (2.1). Equivalently, we want to study the integer roots of the exponential polynomial $\sum_{i=1}^k f_i(z) \alpha_i^z$.

The well-known Skolem-Mahler-Lech theorem says that the zero set of any LRS is a finite union of arithmetic progressions, and furthermore it is a finite set if the sequence is non-degenerate; for example see [9, Theorem 2.1]. There are rich results on the quantitative version of this theorem. Here we restate some results in the setting of exponential polynomials, which are used later on.

In the rest of this section, we fix an exponential polynomial over \mathbb{C}

$$(2.2) \quad F(z) = \sum_{i=1}^k f_i(z) \alpha_i^z$$

with distinct $\alpha_1, \dots, \alpha_k \in \mathbb{C}^*$, and non-zero $f_i \in \mathbb{C}[z]$ for $1 \leq i \leq k$. We also define

$$m = \deg f_1 + \dots + \deg f_k + k$$

and denote

$$\mathcal{Z}(F) = \{n \in \mathbb{Z} : F(n) = 0, n \geq 0\}.$$

Note that $F(z)$ corresponds to an LRS of order m , and the set $\mathcal{Z}(F)$ is exactly the zero set of the corresponding sequence. Especially, when f_1, \dots, f_k are constants, $F(z)$ corresponds to a simple LRS.

The following result comes from [1, Corollary 6.3] and [2, Theorem 1.1].

Lemma 2.1. *Let $F(z)$ be given by (2.2). Then the set $\mathcal{Z}(F)$ is the union of at most $\exp(\exp(70m))$ arithmetic progressions. Moreover, if f_1, \dots, f_k are non-zero constants, then the set $\mathcal{Z}(F)$ is the union of at most $(8m)^{4m^5}$ arithmetic progressions.*

Lemma 2.1 can yield some quantitative results concerning Conjecture 1.1. However, here we are more interested with the case when the subset of integers in Conjecture 1.1 is a finite set.

As mentioned above, if $F(z)$ corresponds to a non-degenerate LRS, the set $\mathcal{Z}(F)$ is in fact a finite set, and furthermore we can bound the cardinality $|\mathcal{Z}(F)|$. The following result follows from [1, Corollary 6.3] and [2, Theorem 1.2].

Lemma 2.2. *Let $F(z)$ be given by (2.2). Suppose that $F(z)$ corresponds to a non-degenerate LRS, and $\deg f_i + 1 \leq a$ for $1 \leq i \leq k$. Then we have*

$$|\mathcal{Z}(F)| \leq (8k^a)^{8k^{6a}};$$

furthermore if f_1, \dots, f_k are non-zero constants, then we have

$$|\mathcal{Z}(F)| \leq (8m)^{4m^5}.$$

In fact, if there exists some index i such that the ratio α_i/α_j is not a root of unity for any $j \neq i$, then the set $\mathcal{Z}(F)$ is still a finite set; see [9, Theorem 2.1 (iii)]. Here we want to bound $|\mathcal{Z}(F)|$ in this case by using Lemma 2.2 and following the arguments in [9].

Lemma 2.3. *Let $F(z)$ be given by (2.2). Let D be the order of the group of roots of unity generated by all those roots of unity which are of the form α_i/α_j for some $1 \leq i, j \leq k$. Suppose that there exists some index i_0 such that the ratio α_{i_0}/α_j is not a root of unity for any $j \neq i_0$, and $\deg f_i + 1 \leq a$ for $1 \leq i \leq k$. Then we have*

$$|\mathcal{Z}(F)| \leq D (8k^a)^{8k^{6a}}.$$

Proof. We partition $\alpha_1, \dots, \alpha_k$ into equivalence classes according to the equivalence relation where $b \sim c$ if and only if the ratio b/c is a root of unity. By renumbering, we can assume that $\alpha_1, \dots, \alpha_s$ are representatives of these equivalence classes. Then, fixing an integer b with $0 \leq b < D$, we consider the equation

$$F(b + nD) = 0$$

with integer unknown $n \geq 0$. By the choice of D , we can express $F(b + nD)$ as

$$F(b + nD) = \sum_{i=1}^s g_i(n) (\alpha_i^D)^n$$

for some polynomials $g_i \in \mathbb{C}[z]$ with $\deg g_i + 1 \leq a$. Under the assumption on α_{i_0} , there indeed exists some index j such that $g_j \neq 0$. So, using Lemma 2.2, we deduce that the cardinality of the set $\{n \geq 0 : F(b + nD) = 0\}$ is at most $(8k^a)^{8k^{6a}}$. The final result follows from the fact that there are D choices of the integer b . \square

Note that if $F(z)$ corresponds to a non-degenerate sequence, then $D = 1$ and Lemma 2.3 is exactly the first upper bound in Lemma 2.2.

Moreover, if $\alpha_1, \dots, \alpha_k$ are roots of a polynomial $f(X)$ over a number field K , then the quantity D can be bounded by

$$(2.3) \quad D < \exp \left(\left(1.05314 + \sqrt{6d} \right) \sqrt{m \log(dm)} \right),$$

where $d = [K : \mathbb{Q}]$ and $m = \deg f \geq 2$; see [12, Theorem 1].

Except for studying the set $\mathcal{Z}(F)$, we also need to estimate the number of integers $n \geq 0$ such that $F(n)$ is equal to a fixed non-zero complex number.

Corollary 2.4. *Let $F(z)$ be given by (2.2). Define $\alpha_{k+1} = 1$. Suppose that there exists some index i_0 such that the ratio α_{i_0}/α_j is not a root of unity for any $j \neq i_0$ with $1 \leq j \leq k+1$. Let D be the order of the group of roots of unity generated by all those roots of unity which are of the form α_i/α_j for some $1 \leq i, j \leq k+1$. Assume that $\deg f_i + 1 \leq a$ for $1 \leq i \leq k$. Then for any $\mu \in \mathbb{C}$ with $\mu \neq 0$, we have*

$$|\{n \in \mathbb{Z} : F(n) = \mu, n \geq 0\}| \leq D (8(k+1)^a)^{8(k+1)^{6a}};$$

furthermore if $F(z)$ corresponds to a non-degenerate LRS, no α_i ($1 \leq i \leq k$) is a root of unity, and f_1, \dots, f_r are non-zero constants, we have

$$|\{n \in \mathbb{Z} : F(n) = \mu, n \geq 0\}| \leq (8(m+1))^{4(m+1)^5}.$$

Proof. Under the assumptions, we can get the desired results by applying Lemmas 2.2 and 2.3 to the following equation

$$\sum_{i=1}^k f_i(n) \alpha_i^n + (-\mu) \cdot 1 = 0, \quad n \geq 0,$$

with coefficients $f_1(n), \dots, f_k(n), -\mu$. \square

When $\alpha_1, \dots, \alpha_k$ are algebraic numbers, the results in Lemma 2.3 and Corollary 2.4 can be improved in some sense. The following lemma is derived from [19, Theorem 1] with a slight refinement. Although the arguments in [19] were presented only for non-degenerate sequences, they are also valid for more general cases under minor changes.

Lemma 2.5. *Let $F(z)$ be given by (2.2). Suppose that $\alpha_1, \dots, \alpha_k$ are algebraic numbers, and let D be the order of the group of roots of unity generated by all those roots of unity which are of the form α_i/α_j for some $1 \leq i, j \leq k$. Denote $K = \mathbb{Q}(\alpha_1, \dots, \alpha_k)$, and assume that $f_1, \dots, f_k \in K[z]$. Let $d = [K : \mathbb{Q}]$, and let ω be the number of prime ideals occurring in the decomposition of the fractional ideals $\langle \alpha_i \rangle$ in K . Assume that there exists some index i_0 such that the ratio α_{i_0}/α_j is not a root of unity for any $j \neq i_0$. Then, we have*

$$|\mathcal{Z}(F)| < D(4(d + \omega))^{2(d+1)}(m - 1);$$

furthermore if K/\mathbb{Q} is a Galois extension but not a cyclic extension, we have

$$|\mathcal{Z}(F)| < D(4(d + \omega))^{d+2}(m - 1).$$

Proof. Here, we sketch the proof for the convenience of the reader.

We first choose a rational prime p such that none of the prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_\omega$ from the decomposition in K of the ideals (α_i) ($1 \leq i \leq k$) divides the ideal $\langle p \rangle$. Let \mathfrak{p} be a prime ideal of K lying above p , and let f denote the residue class degree of $K_{\mathfrak{p}}$ over \mathbb{Q}_p , where \mathbb{Q}_p is the p -adic completion of \mathbb{Q} and $K_{\mathfrak{p}}$ is the completion of K with respect to \mathfrak{p} . Let \mathbb{C}_p be the completion of the algebraic closure of \mathbb{Q}_p . We denote the valuation of \mathbb{C}_p by $|\cdot|_p$, normalised such that $|p|_p = p^{-1}$. Note that $\mathbb{Q}_p \subseteq K_{\mathfrak{p}} \subseteq \mathbb{C}_p$.

By the choice of p , we have

$$|\alpha_i|_p = 1, \quad i = 1, \dots, k.$$

Furthermore, by [19, Equation (3.4)] we know

$$\left| \alpha_i^{p^f - 1} - 1 \right|_p < p^{-1/(p(p-1)) - 1/(p-1)}, \quad i = 1, \dots, k.$$

Then, we have

$$\left| \alpha_i^{D(p^f-1)} - 1 \right|_p \leq \left| \alpha_i^{p^f-1} - 1 \right|_p < p^{-1/(p(p-1))-1/(p-1)}, \quad i = 1, \dots, k.$$

Fix an integer a with $0 \leq a < D(p^f - 1)$, we consider the equation

$$F(a + nD(p^f - 1)) = 0$$

with integer unknown $n \geq 0$.

As in the proof of Lemma 2.3, we partition $\alpha_1, \dots, \alpha_k$ into equivalence classes, and we assume that $\alpha_1, \dots, \alpha_s$ are representatives of these equivalence classes. Then, by the choice of D , we can express $F(a + nD(p^f - 1))$ as

$$F(a + nD(p^f - 1)) = \sum_{i=1}^s g_i(n) \left(\alpha_i^{D(p^f-1)} \right)^n$$

for some polynomials $g_i \in K[z]$. Under the assumption of α_{i_0} , there indeed exists some index j such that $g_j \neq 0$.

As solving the equation (3.6) of [19], we immediately see that the cardinality of the set $\{n \in \mathbb{Z} : F(a + nD(p^f - 1)) = 0, n \geq 0\}$ is at most $(m-1)(p+1)$. Thus, we obtain

$$|\mathcal{Z}(F)| \leq D(p^f - 1)(m-1)(p+1).$$

From [19, Section 4], the prime p can be chosen such that

$$p < (4(d + \omega))^2.$$

Then, the first desired upper bound follows from the fact that $f \leq d$.

Now, we assume that K/\mathbb{Q} is a Galois extension but not a cyclic extension. In order to prove the second claimed upper bound, it suffices to show that p does not remain inert in K . Because if this is true, then $f \leq d/2$, which can conclude the proof.

Let D_p denote the decomposition group of p in K/\mathbb{Q} . Suppose that p remains inert in K . Then, $f = d$, and D_p is a cyclic group of order d . Since $[K : \mathbb{Q}] = d$, D_p is exactly the Galois group $\text{Gal}(K/\mathbb{Q})$. So, K/\mathbb{Q} is a cyclic extension, this leads to a contradiction. \square

Applying the same arguments as in the proof of Corollary 2.4, we can obtain similar results as Lemma 2.5 for the cardinality $|\{n \in \mathbb{Z} : F(n) = \mu, n \geq 0\}|$, where μ is a non-zero algebraic number.

We also need a result on solutions of linear equations in several variables. The following result is given in [2, Lemma 2.1] and is derived from [1, Theorem 6.2].

Lemma 2.6. *Let Γ be finitely generated subgroup of $(\mathbb{C}^*)^k$ of rank r , and let $a_1, \dots, a_k \in \mathbb{C}^*$. Then, up to proportionality, the equation*

$$(2.4) \quad a_1 x_1 + \dots + a_k x_k = 0$$

has less than $(8k)^{4(k-1)^4(k+r)}$ non-degenerate solutions in Γ .

Here, “up to proportionality” means that two solutions (x_1, \dots, x_k) and (y_1, \dots, y_k) of (2.4) are equivalent if there is some non-zero c such that

$$(y_1, \dots, y_k) = (cx_1, \dots, cx_k).$$

Besides, we call a solution of (2.4) *non-degenerate* if no subsum of the left hand side of the equation vanishes.

Let K be a number field, let S be a finite set of places of K containing all the Archimedean places and write \mathcal{O}_S^* for the group of S -units of K . If the coefficients $a_1, \dots, a_k \in K \setminus \{0\}$, then the number of such solutions of (2.4) in $\Gamma \subseteq (\mathcal{O}_S^*)^k$ can be bounded better than Lemma 2.6; for example see [13, Theorem 3]. So, some results in this paper can be improved in this case.

Let \mathcal{P} be a partition of the set $I = \{1, \dots, k\}$. The subsets $\lambda \subseteq I$ occurring in the partition \mathcal{P} are considered as elements of \mathcal{P} . Then, the system of equations

$$(2.4 \mathcal{P}) \quad \sum_{i \in \lambda} a_i x_i = 0 \quad (\lambda \in \mathcal{P})$$

is a refinement of (2.4). Given a partition \mathcal{P} of the set I , we say that two solutions (x_1, \dots, x_k) and (y_1, \dots, y_k) of (2.4) are equivalent *up to proportionality with respect to \mathcal{P}* if both of them are also solutions of the system (2.4 \mathcal{P}), and for each $\lambda \in \mathcal{P}$ the two solutions $(x_i)_{i \in \lambda}$ and $(y_i)_{i \in \lambda}$ of the corresponding equation are equivalent up to proportionality.

Finally, two solutions (x_1, \dots, x_k) and (y_1, \dots, y_k) of (2.4) are called equivalent *up to weak proportionality* if there exists a partition \mathcal{P} of the set I such that they are equivalent up to proportionality with respect to \mathcal{P} .

Now, we want to count all the solutions of (2.4) up to weak proportionality.

Corollary 2.7. *Let Γ be finitely generated subgroup of $(\mathbb{C}^*)^k$ of rank r , and let $a_1, \dots, a_k \in \mathbb{C}^*$. Then, up to weak proportionality, the equation (2.4) has less than*

$$(0.5k)^k (8k)^{4(k-1)^4(k+r)}$$

solutions in Γ .

Proof. Let \mathcal{P} be a partition of the set $I = \{1, \dots, k\}$. Note that in order to ensure that the system (2.4 \mathcal{P}) has solutions in Γ we must have that $|\lambda| \geq 2$ for any $\lambda \in \mathcal{P}$. So, we can assume that $|\mathcal{P}| \leq k/2$.

If \mathcal{Q} is another partition of I such that \mathcal{Q} is a refinement of \mathcal{P} , then the system (2.4 \mathcal{Q}) implies (2.4 \mathcal{P}). Let $\mathcal{T}(\mathcal{P})$ consist of solutions of (2.4 \mathcal{P}) in Γ up to proportionality with respect to \mathcal{P} , which are not solutions of any (2.4 \mathcal{Q}) where \mathcal{Q} is a proper refinement of \mathcal{P} .

According to the partition \mathcal{P} , we can treat Γ as a direct product

$$\Gamma = \prod_{\lambda \in \mathcal{P}} \Gamma(\lambda),$$

where $\Gamma(\lambda)$ is the projection of Γ corresponding to λ . For each $\lambda \in \Gamma$, $\Gamma(\lambda)$ is also a finitely generated group and let $r(\lambda)$ be its rank. Obviously, we have

$$\sum_{\lambda \in \mathcal{P}} r(\lambda) = r.$$

For each equation in (2.4 \mathcal{P})

$$\sum_{i \in \lambda} a_i x_i = 0,$$

by Lemma 2.6 it has less than $(8|\lambda|)^{4(|\lambda|-1)^4(|\lambda|+r(\lambda))}$ non-degenerate solutions in $\Gamma(\lambda)$ up to proportionality. Thus, we have

$$\begin{aligned} |\mathcal{T}(\mathcal{P})| &< \prod_{\lambda \in \mathcal{P}} (8|\lambda|)^{4(|\lambda|-1)^4(|\lambda|+r(\lambda))} \\ &< (8k)^{4(k-1)^4(k+r)}. \end{aligned}$$

Recall that the Bell numbers count the number of partitions of a set. By [8, Theorem 2.1], the number of partitions of I is less than

$$(0.792k / \log(k+1))^k.$$

However, not all such partitions are suitable for our settings. We have indicated that a *suitable partition* \mathcal{P} should satisfy that $|\lambda| \geq 2$ for any $\lambda \in \mathcal{P}$. So, the number of these suitable partitions is not greater than $(0.5k)^k$.

Note that every solution of the equation (2.4) is a solution of the system (2.4 \mathcal{P}) for some partition \mathcal{P} , and we can assume that $k \geq 2$. So, up to weak proportionality, the number of solutions of (2.4) in Γ is at most

$$\sum_{\mathcal{P}} |\mathcal{T}(\mathcal{P})| < (0.5k)^k (8k)^{4(k-1)^4(k+r)},$$

where the sum runs through all suitable partitions of I . This completes the proof. \square

Finally, we state a result due to Schlickewei and Schmidt [20, Theorem 1]. For a vector $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{Z}^m$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in \overline{\mathbb{Q}}^m$, we denote

$$\boldsymbol{\alpha}^{\mathbf{x}} = \alpha_1^{x_1} \cdots \alpha_m^{x_m}.$$

Lemma 2.8. *Fix $\boldsymbol{\alpha}_i \in (\overline{\mathbb{Q}}^*)^m$, $i = 1, \dots, k$, such that for any $1 \leq i \neq j \leq k$ the set of $\mathbf{z} \in \mathbb{Z}^m$ with*

$$\boldsymbol{\alpha}_i^{\mathbf{z}} = \boldsymbol{\alpha}_j^{\mathbf{z}}$$

contains only the zero vector. Then, the number of solutions to the equation

$$\sum_{i=1}^k a_i \boldsymbol{\alpha}_i^{\mathbf{x}} = 0$$

with non-zero algebraic numbers a_i is less than

$$(0.5k)^k 2^{35B^3} d^{6B^2},$$

where $B = \max(m, k)$ and d is the degree of the number field generated by the coefficients a_i and the vectors $\boldsymbol{\alpha}_i$.

Proof. The desired result can be easily proved by using [20, Theorem 1] and counting the solutions through all the partitions of the set $\{1, 2, \dots, k\}$. \square

3. MAIN RESULTS

We first recall a notation. For any polynomial $G \in \mathbb{C}[X_1, \dots, X_m]$, we let

$$\mathbf{n}(G) = \text{the number of monomials of } G.$$

As usual, here we treat the non-zero constant term as a monomial.

Recall that for any point $\mathbf{w} \in \mathbb{C}^m$, we denote its coordinates by (w_1, \dots, w_m) . We start the discussions by dealing with the simplest case.

Theorem 3.1. *Let $\Phi = (X_1^d, \dots, X_m^d)$ with integer $d \geq 2$. Fix a hypersurface $V = Z(G)$, where*

$$G = \sum_{i_1, \dots, i_m} a_{i_1, \dots, i_m} X_1^{i_1} \cdots X_m^{i_m} \in \mathbb{C}[X_1, \dots, X_m]$$

with $G \neq 0$ and $i_j \geq 0$ for $1 \leq j \leq m$. Then, for any $\mathbf{w} \in (\mathbb{C}^)^m$ with multiplicatively independent coordinates, we have*

$$|\mathcal{S}_{\mathbf{w}}(\Phi, V)| \leq (8\mathbf{n}(G))^{4\mathbf{n}(G)^5}.$$

Proof. For the given point \mathbf{w} , we want to bound the number of integers $n \geq 0$ such that $\Phi^{(n)}(\mathbf{w}) \in V$, that is

$$\sum_{i_1, \dots, i_m} a_{i_1, \dots, i_m} (w_1^{i_1} \cdots w_m^{i_m})^{d^n} = 0.$$

This is upper bounded by the number of integers $n \geq 0$ such that

$$\sum_{i_1, \dots, i_m} a_{i_1, \dots, i_m} (w_1^{i_1} \cdots w_m^{i_m})^n = 0.$$

Since the coordinates of \mathbf{w} are multiplicatively independent, the upper bound follows from Lemma 2.2 and Corollary 2.4 by noticing whether G has a non-zero constant term or not. \square

One can relax the multiplicative independence condition on the point \mathbf{w} in some special cases.

Theorem 3.2. *Let $\Phi = (X_1^d, \dots, X_m^d)$ with integer $d \geq 2$. Fix a hypersurface $V = Z(G)$, where*

$$G = \sum_{j=1}^m a_j X_j^{e_j} \in \mathbb{C}[X_1, \dots, X_m]$$

with $G \neq 0$ and $e_j \geq 0$ for $1 \leq j \leq m$. For any point $\mathbf{w} \in \mathbb{C}^m$, let D be the order of the group of roots of unity generated by all those roots of unity which are of the form w_i/w_j for some $1 \leq i, j \leq m$. Suppose that there exists some index j_0 such that w_{j_0} is not a root of unity, $w_{j_0} \neq 0, a_{j_0} \neq 0, e_{j_0} \neq 0$ and the ratio w_j/w_{j_0} is not a root of unity for any $j \neq j_0$. Then, we have

$$|\mathcal{S}_{\mathbf{w}}(\Phi, V)| \leq D(8\mathfrak{n}(G))^{\mathfrak{sn}(G)^6}.$$

Proof. Applying the same arguments as in the proof of Theorem 3.1, the desired result follows directly from Lemma 2.3 and Corollary 2.4. \square

The upper bound in Theorem 3.2 is not uniform because of the quantity D . However, we can make it uniform in some sense. In fact, if we choose the point $\mathbf{w} \in K^m$, where K is a number field, then D does not exceed the number of roots of unity contained in K . Alternatively, one can also use (2.3).

We also want to indicate that in Theorem 3.2 if we further assume that $\mathbf{w} \in \overline{\mathbb{Q}}^m$ and $G \in \overline{\mathbb{Q}}[X_1, \dots, X_m]$, then under the same assumptions as in Lemma 2.5, we can get another upper bound for $|\mathcal{S}_{\mathbf{w}}(\Phi, V)|$.

Obtaining results on the size of $\mathcal{S}_{\mathbf{w}}(\Phi, V)$ even for the slightly more general case when $F_i = X_i^{d_i}$, $i = 1, \dots, m$, where the degrees d_i are not necessarily the same, seems not to be quite straightforward.

Theorem 3.3. *Let $\Phi = (X_1^{d_1}, \dots, X_m^{d_m})$ with integers $d_i \geq 2$ ($1 \leq i \leq m$). Fix a hypersurface $V = Z(G)$, where*

$$G = \sum_{i_1, \dots, i_m} a_{i_1, \dots, i_m} X_1^{i_1} \cdots X_m^{i_m} \in \mathbb{C}[X_1, \dots, X_m]$$

with $G \neq 0$ and $i_j \geq 0$ for $1 \leq j \leq m$. Then, for any $\mathbf{w} \in (\mathbb{C}^)^m$ with multiplicatively independent coordinates, we have*

$$|\mathcal{S}_{\mathbf{w}}(\Phi, V)| \leq (0.5\mathbf{n}(G))^{\mathbf{n}(G)} (8\mathbf{n}(G))^{4\mathbf{n}(G)(\mathbf{n}(G)-1)^4(m+1)}.$$

Proof. Since the polynomial G has $\mathbf{n}(G)$ monomials, we renumber the indices (i_1, \dots, i_m) as $1, 2, \dots, \mathbf{n}(G)$. If the index (i_1, \dots, i_m) corresponds to j ($1 \leq j \leq \mathbf{n}(G)$), then accordingly we write a_{i_1, \dots, i_m} and $X_1^{i_1} \cdots X_m^{i_m}$ as b_j and Y_j , respectively. So, we have

$$G = \sum_{j=1}^{\mathbf{n}(G)} b_j Y_j.$$

For the given point \mathbf{w} , bounding $|\mathcal{S}_{\mathbf{w}}(\Phi, V)|$ is exactly to bound the number of integers $n \geq 0$ such that $\Phi^{(n)}(\mathbf{w}) \in V$, that is,

$$(3.1) \quad \sum_{j=1}^{\mathbf{n}(G)} b_j Y_j (\Phi^{(n)}(\mathbf{w})) = 0.$$

Let Λ be the group generated by the coordinates of \mathbf{w} , and let $\Gamma = \Lambda^{\mathbf{n}(G)}$. Since the rank of Λ is m , the rank of Γ is at most $m\mathbf{n}(G)$. In view of (3.1), we consider the solutions $(x_1, \dots, x_{\mathbf{n}(G)})$ of the equation

$$(3.2) \quad b_1 x_1 + \cdots + b_{\mathbf{n}(G)} x_{\mathbf{n}(G)} = 0$$

in Γ . By Corollary 2.7, the equation (3.2) has less than

$$(0.5\mathbf{n}(G))^{\mathbf{n}(G)} (8\mathbf{n}(G))^{4(\mathbf{n}(G)-1)^4(\mathbf{n}(G)+m\mathbf{n}(G))}$$

solutions in Γ up to weak proportionality.

For any $1 \leq i \neq j \leq \mathbf{n}(G)$, write $Y_i = X_1^{i_1} \cdots X_m^{i_m}$ and $Y_j = X_1^{j_1} \cdots X_m^{j_m}$. Suppose that there exist integers n, k with $0 \leq n < k$ such that

$$\frac{Y_i(\Phi^{(k)}(\mathbf{w}))}{Y_i(\Phi^{(n)}(\mathbf{w}))} = \frac{Y_j(\Phi^{(k)}(\mathbf{w}))}{Y_j(\Phi^{(n)}(\mathbf{w}))}.$$

Then due to $\Phi^{(n)} = (X_1^{d_1^n}, \dots, X_m^{d_m^n})$, we obtain

$$w_1^{i_1(d_1^k - d_1^n)} \cdots w_m^{i_m(d_m^k - d_m^n)} = w_1^{j_1(d_1^k - d_1^n)} \cdots w_m^{j_m(d_m^k - d_m^n)}.$$

Noticing that $d_\ell \geq 2$ ($1 \leq \ell \leq m$) and the coordinates w_1, \dots, w_m are multiplicatively independent, we must have $i_1 = j_1, \dots, i_m = j_m$, which implies that $Y_i = Y_j$. This is a contradiction with $Y_i \neq Y_j$.

Hence, for any $1 \leq i \neq j \leq \mathbf{n}(G)$ and any $0 \leq n < k$, we have

$$\frac{Y_i(\Phi^{(k)}(\mathbf{w}))}{Y_i(\Phi^{(n)}(\mathbf{w}))} \neq \frac{Y_j(\Phi^{(k)}(\mathbf{w}))}{Y_j(\Phi^{(n)}(\mathbf{w}))}.$$

Thus, the number of those solutions of (3.2) with the form

$$(3.3) \quad (Y_1(\Phi^{(n)}(\mathbf{w})), \dots, Y_{\mathbf{n}(G)}(\Phi^{(n)}(\mathbf{w}))) \quad (n = 0, 1, 2, \dots)$$

is less than

$$(0.5\mathbf{n}(G))^{\mathbf{n}(G)}(8\mathbf{n}(G))^{4\mathbf{n}(G)(\mathbf{n}(G)-1)^4(m+1)}.$$

Notice that there is a one-to-one correspondence between the vectors (3.3) and the integers $n \geq 0$, we complete the proof. \square

Now, we want to use Lemma 2.8 to give another method on handling a special case in Theorem 3.3.

Theorem 3.4. *Let K be a number field of degree d . Let the system $\Phi = (X_1^{d_1}, \dots, X_m^{d_m})$ with integers $d_i \geq 0$ ($1 \leq i \leq m$) and some index ℓ such that $d_\ell \geq 2$. Fix a hypersurface $V = Z(G)$, where*

$$G = \sum_{i_1, \dots, i_m} a_{i_1, \dots, i_m} X_1^{i_1} \cdots X_m^{i_m} \in K[X_1, \dots, X_m]$$

with $G \neq 0$ and $i_j \geq 0$ for $1 \leq j \leq m$, such that G has zero constant term. Suppose that for any two monomials $a_{i_1, \dots, i_m} X_1^{i_1} \cdots X_m^{i_m}$ and $a_{j_1, \dots, j_m} X_1^{j_1} \cdots X_m^{j_m}$ of G , we have $i_1 \neq j_1, \dots, i_m \neq j_m$. Then, for any $\mathbf{w} \in (K^*)^m$ with multiplicatively independent coordinates, we have

$$|\mathcal{S}_{\mathbf{w}}(\Phi, V)| \leq (0.5\mathbf{n}(G))^{\mathbf{n}(G)} 2^{35B^3} d^{6B^2},$$

where $B = \max(m, \mathbf{n}(G))$.

Proof. Since $\Phi^{(n)} = (X_1^{d_1^n}, \dots, X_m^{d_m^n})$, we need to bound the number of integers $n \geq 0$ such that

$$\sum_{i_1, \dots, i_m} a_{i_1, \dots, i_m} w_1^{i_1 d_1^n} \cdots w_m^{i_m d_m^n} = 0.$$

For each index (i_1, \dots, i_m) , we write $\alpha_i = (w_1^{i_1}, \dots, w_m^{i_m})$ and $\mathbf{x} = (d_1^n, \dots, d_m^n)$, then the above equation becomes

$$(3.4) \quad \sum_{i_1, \dots, i_m} a_{i_1, \dots, i_m} \alpha_i^{\mathbf{x}} = 0.$$

Moreover, for any two indices (i_1, \dots, i_m) and (j_1, \dots, j_m) , if $\alpha_i^{\mathbf{z}} = \alpha_j^{\mathbf{z}}$ for $\mathbf{z} = (z_1, \dots, z_m) \in \mathbb{Z}^m$, then, we must have $i_1 z_1 = j_1 z_1, \dots, i_m z_m = j_m z_m$ by using the multiplicative independence condition. Under the assumption that $i_1 \neq j_1, \dots, i_m \neq j_m$, we get $z_1 = \dots = z_m = 0$. That is, \mathbf{z} is the zero vector.

Now, applying Lemma 2.8 to the equation (3.4) we know that the equation (3.4) has less than

$$(0.5\mathbf{n}(G))^{\mathbf{n}(G)} 2^{35B^3} d^{6B^2}$$

solutions with the form (d_1^n, \dots, d_m^n) , $n = 0, 1, 2, \dots$. Besides, since $d_\ell \geq 2$, we have $(d_1^n, \dots, d_m^n) \neq (d_1^k, \dots, d_m^k)$ for any integers $n \neq k$. So, the desired result follows. \square

It seems natural to expect that the classes of dynamical systems and hypersurfaces that satisfy the uniform boundedness condition are quite wide. We confirm this by the following three theorems.

Theorem 3.5. *Let $\Phi = (X_1^d, F_2, F_3, \dots, F_m)$ with integer $d \geq 2$, where*

$$F_i = X_1^{s_{i1}} \dots X_m^{s_{im}}$$

with $s_{ij} \geq 0, s_{ii} \geq 1, 2 \leq i \leq m, 1 \leq j \leq m$, such that $\deg F_i < d$ for any $2 \leq i \leq m$. Fix a hypersurface $V = Z(G)$, where

$$G = aX_1^e + \sum_{i_1, \dots, i_m} a_{i_1, \dots, i_m} X_1^{i_1} \dots X_m^{i_m} \in \mathbb{C}[X_1, \dots, X_m]$$

with $a \neq 0, e \geq 1$ and $i_j \geq 0$ for $1 \leq j \leq m$. Suppose that $\deg G = e$. Then, for any $\mathbf{w} \in (\mathbb{C}^)^m$ with multiplicatively independent coordinates, we have*

$$|\mathcal{S}_{\mathbf{w}}(\Phi, V)| \leq (0.5\mathbf{n}(G))^{\mathbf{n}(G)} (8\mathbf{n}(G))^{4\mathbf{n}(G)(\mathbf{n}(G)-1)^4(m+1)}.$$

Proof. As in the proof of Theorem 3.3 and under the assumptions of the polynomial G , we can write

$$G = \sum_{j=1}^{\mathbf{n}(G)} b_j Y_j$$

such that $b_1 = a, Y_1 = X_1^e$. We need to bound the number of integers $n \geq 0$ such that

$$\sum_{j=1}^{\mathbf{n}(G)} b_j Y_j (\Phi^{(n)}(\mathbf{w})) = 0.$$

For any $n \geq 0$, as in Section 1.3 we write $\Phi^{(n)} = (F_1^{(n)}, F_2^{(n)}, \dots, F_m^{(n)})$ with $F_1^{(n)} = X_1^{d^n}$. Since $d > \deg F_i$ for any $2 \leq i \leq m$, we can see that

$$(3.5) \quad \deg F_i^{(n)} < d^n, \quad \text{for any } n \geq 1.$$

Fix $2 \leq i \leq m$. Then, for any $n \geq 0$, since

$$\Phi^{(n+1)} = \Phi(F_1^{(n)}, F_2^{(n)}, \dots, F_m^{(n)}) \quad \text{and} \quad s_{ii} \geq 1,$$

applying (3.5) we deduce that

$$\begin{aligned} & \deg_{X_1} F_i^{(n+1)} - \deg_{X_1} F_i^{(n)} \\ &= \sum_{j=1, j \neq i}^m s_{ij} \deg_{X_1} F_j^{(n)} + (s_{ii} - 1) \deg_{X_1} F_i^{(n)} \\ &< d^n \left(\sum_{j=1}^m s_{ij} - 1 \right) < d^n (d - 1). \end{aligned}$$

So, for any integers n, k with $0 \leq n < k$, we obtain

$$(3.6) \quad 0 \leq \deg_{X_1} F_i^{(k)} - \deg_{X_1} F_i^{(n)} < d^k - d^n.$$

Note that for any integers n, k with $0 \leq n < k$, we have

$$\frac{Y_1(\Phi^{(k)}(\mathbf{w}))}{Y_1(\Phi^{(n)}(\mathbf{w}))} = w_1^{e(d^k - d^n)}.$$

Combining (3.6) with $\deg G = e$, we can see that for any $2 \leq j \leq \mathbf{n}(G)$, the degree of $Y_j(\Phi^{(k)}(\mathbf{w})) / Y_j(\Phi^{(n)}(\mathbf{w}))$ with respect to w_1 is less than $e(d^k - d^n)$. Notice that the coordinates of \mathbf{w} are multiplicatively independent. Thus, for any $2 \leq j \leq \mathbf{n}(G)$ we have

$$\frac{Y_1(\Phi^{(k)}(\mathbf{w}))}{Y_1(\Phi^{(n)}(\mathbf{w}))} \neq \frac{Y_j(\Phi^{(k)}(\mathbf{w}))}{Y_j(\Phi^{(n)}(\mathbf{w}))}.$$

Hence as before, the equation $b_1 x_1 + \dots + b_{\mathbf{n}(G)} x_{\mathbf{n}(G)} = 0$ has less than

$$(0.5\mathbf{n}(G))^{\mathbf{n}(G)} (8\mathbf{n}(G))^{4\mathbf{n}(G)(\mathbf{n}(G)-1)^4(m+1)}$$

solutions with the form

$$(Y_1(\Phi^{(n)}(\mathbf{w})), \dots, Y_{\mathbf{n}(G)}(\Phi^{(n)}(\mathbf{w}))) \quad (n = 0, 1, 2, \dots).$$

Finally, the desired result follows from the one-to-one correspondence between the above vectors and the integers $n \geq 0$. \square

Theorem 3.6. *Let $\Phi = (F_1, \dots, F_m)$ with $m \geq 2$ be defined by*

$$F_i = X_i^{s_i} X_{i+1}^{s_{i+1}} \dots X_m^{s_{im}},$$

with

$$s_i > 1, s_{ij} \geq 0, \quad j = i + 1, \dots, m,$$

or

$$s_i \geq 1, s_{ij} \geq 1 \text{ for at least one } j = i + 1, \dots, m,$$

$i = 1, \dots, m-1$, and

$$F_m = X_m.$$

Fix a hypersurface $V = Z(G)$, where

$$G = \sum_{i_1, \dots, i_m} a_{i_1, \dots, i_m} X_1^{i_1} \cdots X_m^{i_m} \in \mathbb{C}[X_1, \dots, X_m]$$

with only one monomial of the form $cX_m^{e_m}$, $c \in \mathbb{C}^*$, $e_m \geq 1$, such that G has a monomial divisible by X_j for some $1 \leq j \leq m-1$ and zero constant term. Then, for any $\mathbf{w} \in (\mathbb{C}^*)^m$ with multiplicatively independent coordinates, we have

$$|\mathcal{S}_{\mathbf{w}}(\Phi, V)| < (0.5\mathbf{n}(G))^{\mathbf{n}(G)} (8\mathbf{n}(G))^{4\mathbf{n}(G)(\mathbf{n}(G)-1)^4(m+1)}.$$

Proof. As in the proof of Theorem 3.3 and under the assumptions, we can write

$$G = \sum_{j=1}^{\mathbf{n}(G)} b_j Y_j$$

such that $b_1 = c$, $Y_1 = X_m^{e_m}$ and Y_2 is not a constant.

For the given point \mathbf{w} , noticing $F_m^{(n)}(\mathbf{w}) = w_m$ for any $n \geq 0$, we need to bound the number of integers $n \geq 0$ such that

$$b_1 w_m^{e_m} + \sum_{j=2}^{\mathbf{n}(G)} b_j Y_j(\Phi^{(n)}(\mathbf{w})) = 0$$

We first claim that if $n \neq k$, then $Y_2(\Phi^{(n)}(\mathbf{w})) \neq Y_2(\Phi^{(k)}(\mathbf{w}))$. Indeed, assume that $n < k$. We note that, by the conditions on s_i and s_{ij} and the fact that the degree $d_{i,n} = \deg F_i^{(n)}$ satisfies

$$d_{i,n} = s_i d_{i,n-1} + s_{i,i+1} d_{i+1,n-1} + \cdots + s_{i,m-1} d_{m-1,n-1} + s_{im},$$

by induction one easily proves that $\deg F_i^{(n)} < \deg F_i^{(k)}$ for any $1 \leq i \leq m$ (the case $s_i = 1$ for all $i = 1, \dots, m-1$ is also proved in [18, Lemma 1]). Thus, by the multiplicative independence of the coordinates of \mathbf{w} , we deduce that $Y_2(\Phi^{(n)}(\mathbf{w})) \neq Y_2(\Phi^{(k)}(\mathbf{w}))$.

Hence, similar as before the equation $b_1 x_1 + \dots + b_{\mathbf{n}(G)} x_{\mathbf{n}(G)} = 0$ has less than

$$(0.5\mathbf{n}(G))^{\mathbf{n}(G)} (8\mathbf{n}(G))^{4\mathbf{n}(G)(\mathbf{n}(G)-1)^4(m+1)}$$

solutions with the form

$$(w_m^{e_m}, Y_2(\Phi^{(n)}(\mathbf{w})), \dots, Y_{\mathbf{n}(G)}(\Phi^{(n)}(\mathbf{w}))) \quad (n = 0, 1, 2, \dots).$$

Now, the desired result follows as before. \square

Theorem 3.7. *Let $\Phi = (F_1, \dots, F_m)$ be defined by*

$$F_i = \prod_{j=1}^m X_j^{s_{ij}}, \quad s_{ij} \geq 0$$

for $i = 1, \dots, m$ and $j = 1, \dots, m$ such that for any $1 \leq i \leq m$ the degree $\deg F_i \geq 2$. Fix a hypersurface $V = Z(G)$, where

$$G = \sum_{i_1, \dots, i_m} a_{i_1, \dots, i_m} X_1^{i_1} \cdots X_m^{i_m} \in \mathbb{C}[X_1, \dots, X_m]$$

with a non-zero constant term c . Then, for any $\mathbf{w} \in (\mathbb{C}^)^m$ with multiplicatively independent coordinates, we have*

$$|\mathcal{S}_{\mathbf{w}}(\Phi, V)| < (0.5\mathbf{n}(G))^{\mathbf{n}(G)} (8\mathbf{n}(G))^{4\mathbf{n}(G)(\mathbf{n}(G)-1)^4(m+1)}.$$

Proof. As in the proof of Theorem 3.3, we can write

$$G = \sum_{j=1}^{\mathbf{n}(G)} b_j Y_j$$

with $b_1 = c$ and $Y_1 = 1$. What we need is to bound the number of integers $n \geq 0$ such that

$$\sum_{j=1}^{\mathbf{n}(G)} b_j Y_j (\Phi^{(n)}(\mathbf{w})) = 0.$$

Since $\deg F_i \geq 2$ for any $1 \leq i \leq m$, we know that

$$\deg F_i^{(n+1)} > \deg F_i^{(n)}, \quad \text{for any } n \geq 0.$$

So, in view of the multiplicative independence of the coordinates of \mathbf{w} , for any integers n, k with $0 \leq n < k$, we have

$$\frac{Y_j (\Phi^{(k)}(\mathbf{w}))}{Y_j (\Phi^{(n)}(\mathbf{w}))} \neq 1, \quad \text{for any } 2 \leq j \leq \mathbf{n}(G).$$

Thus as before, the equation $b_1 x_1 + \dots + b_{\mathbf{n}(G)} x_{\mathbf{n}(G)} = 0$ has less than

$$(0.5\mathbf{n}(G))^{\mathbf{n}(G)} (8\mathbf{n}(G))^{4\mathbf{n}(G)(\mathbf{n}(G)-1)^4(m+1)}$$

solutions with the form

$$(1, Y_2 (\Phi^{(n)}(\mathbf{w})), \dots, Y_{\mathbf{n}(G)} (\Phi^{(n)}(\mathbf{w}))) \quad (n = 0, 1, 2, \dots).$$

We conclude the proof by noticing the one-to-one correspondence between the above vectors and the integers $n \geq 0$. \square

We remark that the assumption on Φ put in Theorem 3.7 is reasonable. For example, let $m = 2$ and fix such a point \mathbf{w} , choose $\Phi = (X_1, X_2^2)$ and $G = X_1 - w_1$, then for any $n \geq 0$ we have $\Phi^{(n)}(\mathbf{w}) \in Z(G)$.

4. COMMENTS

It is quite sure that one can get more partial results concerning the quantitative dynamical Mordell-Lang conjecture by using the methods presented in Section 3. It is also likely that several upper bounds in Section 3 can be improved in some special cases.

However, the main method based on Corollary 2.7 requires that for each $\mathbf{w} \in (\mathbb{C}^*)^m$, all $\Phi^{(n)}(\mathbf{w})$, $n \geq 1$, are contained in the same finitely generated group Γ of $(\mathbb{C}^*)^m$. Thus, we were able to ensure this property only for monomial systems.

The only non-monomial example for which one can obtain similar results as in Theorems 3.3–3.7 is the following: let $\mathcal{F} = \mathcal{A}^{-1} \circ \Phi \circ \mathcal{A}$, where \mathcal{A} is a polynomial automorphism and Φ is any monomial system defined in the results of Section 3. Then, for any $n \geq 1$, we have

$$\mathcal{F}^{(n)} = \mathcal{A}^{-1} \circ \Phi^{(n)} \circ \mathcal{A}.$$

Thus, for a hypersurface $V = Z(G)$ and a point $\mathbf{w} \in (\mathbb{C}^*)^m$, the point $\mathcal{F}^{(n)}(\mathbf{w}) \in V$ if and only if $\Phi^{(n)}(\mathbf{v}) \in Z(G(\mathcal{A}^{-1}))$, where $\mathbf{v} = \mathcal{A}(\mathbf{w})$. If the coordinates of \mathbf{v} are multiplicatively independent, then one can obtain similar results as in Theorems 3.3–3.7.

It is worth remarking that our methods can also be employed to study the synchronized intersection of two orbits. Indeed, let \mathcal{F} and \mathcal{H} be two polynomial systems from \mathbb{C}^m to itself, then one can ask to bound the size of the subset of integers

$$\{n \geq 0 : \mathcal{F}^{(n)}(\mathbf{w}_1) = \mathcal{H}^{(n)}(\mathbf{w}_2)\},$$

where $\mathbf{w}_1, \mathbf{w}_2$ are two vectors in \mathbb{C}^m . Now, we take $\Phi = (\mathcal{F}, \mathcal{H})$ as in our results (but we see \mathcal{F} in variables X_1, \dots, X_m and \mathcal{H} in variables Y_1, \dots, Y_m , so we have $2m$ variables), and $V = Z(G)$, where $G = \sum_{i=1}^m (X_i - Y_i)$. Clearly, we have

$$\{n \geq 0 : \mathcal{F}^{(n)}(\mathbf{w}_1) = \mathcal{H}^{(n)}(\mathbf{w}_2)\} \subseteq \mathcal{S}_{(\mathbf{w}_1, \mathbf{w}_2)}(\Phi, V),$$

where $\mathcal{S}_{(\mathbf{w}_1, \mathbf{w}_2)}(\Phi, V)$ is defined as in (1.1). So, the problem turns out to bound the size $|\mathcal{S}_{(\mathbf{w}_1, \mathbf{w}_2)}(\Phi, V)|$, which can be done in many cases by applying the methods in Section 3.

In Section 3, we consider uniform boundedness of the size of the intersection $\text{Orb}_{\mathbf{w}}(\Phi) \cap V$ for various cases. One can also consider obtaining upper bounds under which there is indeed an integer n such that $\Phi^{(n)}(\mathbf{w}) \in V$, or which are greater than any integer n with $\Phi^{(n)}(\mathbf{w}) \in V$. For example, in Theorem 3.2, if we further assume that $|w_{j_0}| > |w_j|$ and $e_{j_0} \geq e_j$ for any $j \neq j_0$, and define

$$a = \max_{j \neq j_0} |a_j| \quad \text{and} \quad \text{the index } i \text{ such that } |w_i^{e_i}| = \max_{j \neq j_0} |w_j^{e_j}|,$$

then we can easily get a lower bound of n from the following inequality

$$\left| a_{j_0} w_{j_0}^{e_{j_0} d^n} \right| > (m-1)a \left| w_i^{e_i d^n} \right|$$

such that this bound is larger than any integer n with $\Phi^{(n)}(\mathbf{w}) \in V$.

We also note that if one imposes in Theorem 3.1 that instead of the multiplicative independence of the coordinates of $\mathbf{w} \in \mathbb{C}^m$, the absolute values of these coordinates are multiplicatively independent, then there exists $I_0 = (i_1, \dots, i_m)$ such that $X_1^{i_1} \cdots X_m^{i_m}$ is a monomial of G and

$$|\mathbf{w}^{I_0}| > |\mathbf{w}^I|$$

with $I = (j_1, \dots, j_m)$ for any other monomial $X_1^{j_1} \cdots X_m^{j_m}$ of G (that is, for $I \neq I_0$). This also allows us to obtain an explicit bound on the largest n such that $\Phi^{(n)}(\mathbf{w}) \in V$.

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